

THE CHINESE UNIVERSITY OF HONG KONG  
DEPARTMENT OF MATHEMATICS  
MATH2050 } First Term  
Mathematical Analysis I  
Homework II

Questions with \* will be marked.

1. \* Let  $A := \{x \in \mathbb{R} : 0 < x \leq 1\}$ , i.e.  $A = (0, 1]$ . Find  $\max A$ ,  $\inf A$ ,  $\max A$  and  $\sup A$  (if exist).  
Give your reasoning (including for your non-existence claim).

2. Do the same for the set

$$S := \left\{ \frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N} \right\}.$$

3. \* Let  $f, g$  be real-valued bounded above functions on a set  $X$ . Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\},$$

or, in convenient notations

$$\sup_{x \in X} (f(x) + g(x)) \leq \sup_{x \in X} f(x) + \sup_{x \in X} g(x).$$

Can the strict inequality (or equality) happen? *Do similar questions for inf.*

4. Let  $(x_n)$  be a sequence converge to a real number  $x$ . Show by  $(\varepsilon - N)$  definition, that

(a)  $\lim_{n \rightarrow \infty} |x_n| = |x|$ ;

(b) if  $\alpha < x < \beta$  then there exists  $N \in \mathbb{N}$  such that  $\alpha < x_n < \beta$  for all  $n \geq N$ . (Hint: consider  $\varepsilon_0 := \min\{\beta - x, x - \alpha\}$ .)

5. Let  $z < x < y$  and  $0 < l < y - x$ . Show that there exists  $\bar{m} \in \mathbb{N}$  such that  $x < z + \bar{m}l < y$ . *Hint: Let  $A := \{n \in \mathbb{N} \setminus \{0\} : z + nl \leq x\}$  and  $\bar{n} = \max A$  (why exists?). Check  $\bar{m} = \bar{n} + 1$  has the desired properties.*

6. \* (Density of  $\mathbb{Q}$ ). Let  $x < y$  be real numbers and let  $n \in \mathbb{N}$  be s.t.  $\frac{1}{n} < y - x$ . Let

$$A := \left\{ k \in \mathbb{Z} : \frac{k}{n} \leq x \right\}.$$

Show that  $A$  is non-empty and is bounded above. By what result (how to state the result),  $A$  has the largest element (to be divided by  $\bar{k}$ ) and hence find  $r \in \mathbb{Q}$  s.t.  $x < r < y$ .  
2nd Method. Apply Q 5 to provide a solution for Q 6.